Увод в теорията на операторни алгебри с приложение в математиката и физиката
Introduction to Theory of Operator Algebras with Applications to Mathematics and Physics
Nikolay M. Nikolov
Lecture 5

Gelfand Theory - part 2

Reminder:

For $\mathcal{A}$ - unital commutative algebra:

$X(\mathcal{A}) := \{ \chi \mid \chi$ is a character, i.e. $\chi : \mathcal{A} \to \mathbb{C}$ is a morphism of unital algebras$\}$

Gelfand Map: $\quad \mathcal{A} \to \mathrm{Maps}\,(X(\mathcal{A}), \mathbb{C})$

$\qquad a \mapsto \hat{a} : \hat{a}(\chi) := \chi(a)$

For unital commutative Banach algebra $\hat{a}\big(X(\mathcal{A})\big) = \sigma_{\mathcal{A}}(a)$,

$\Rightarrow \quad \hat{a} \in$ Bonded Maps $\big(X(\mathcal{A}), \mathbb{C}\big)$.

Kernel of the Gelfand Map $= \bigcap \{ \mathcal{J}' \mid \mathcal{J}' \subseteq \mathcal{A}$ is a maximal ideal$\}$

$= 0$ - for unital commutative C^*-algeba

"Proposition 4.13" If $\mathcal{A}$ is unital commutative C^*-algeba generated by $\underbrace{a = a^*}$ and 1, then $\mathcal{A} \cong C\big(\sigma_{\mathcal{A}}(a)\big)$

$a \mapsto p(t) = t \,|_{\sigma_{\mathcal{A}}(a)}$

what is the role of this condition?

As a corollary: $\sigma_{\mathcal{A}}(a^*a) \subseteq [0,\infty)$ for $a \in \mathcal{A}$ - unital C^*-algebra

Since: $\sigma_{\mathcal{A}}(a^*a) = \widehat{a^*a}\,\big(X(\mathcal{A})\big) = \{ \widehat{a^*a}(\chi) = \chi(a^*a) \mid \chi \in X(\mathcal{A})\}$

$= \{ \chi(a^*)\,\chi(a) = \overline{\chi(a)}\,\chi(a) = |\chi(a)|^2 \mid \chi \in X(\mathcal{A})\}$

$\subseteq [0,\infty)$

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Prime ideals (прости идеали) in commutative associative algebras

<u>Def.</u> A prime ideal $\mathcal{J}$ in a commutative associative algebra $\mathcal{A}$ is a proper ideal with the property: $a \cdot b \in \mathcal{J} \Rightarrow a \in \mathcal{J}$ or $b \in \mathcal{J}$ $(\forall a, b \in \mathcal{A})$.
Zero divisor $a \in \mathcal{A}$ is such $\neq 0$ element that $a \cdot b = 0$ for some $b \in \mathcal{A}$.

<u>Observation 5.1</u> A proper ideal $\mathcal{J}$ in a commutative associative algebra $\mathcal{A}$ is prime iff $(\Leftrightarrow)$ $\mathcal{A}/\mathcal{J}$ has no zero divisors (= integral domain област на цялостност).

<u>Proof.</u> $[a \cdot b] = [a] \cdot [b] = 0 \Leftrightarrow [a \cdot b] = 0 \Leftrightarrow a \cdot b \in \mathcal{J}$. $\square$

<u>Def.</u> A multiplicative system $\mathcal{S}$ in a commutative associative algebra $\mathcal{A}$ is such a subset that $1 \in \mathcal{S}$, $0 \notin \mathcal{S}$, $a, b \in \mathcal{S} \Rightarrow a \cdot b \in \mathcal{S}$.

<u>Theorem</u> Let $\mathcal{S}$ be a multiplicative system in a commutative associative algebra $\mathcal{A}$. In the set Γ of all ideals $\mathcal{J} \subseteq \mathcal{A}$ such that $\mathcal{J} \cap \mathcal{S} = \emptyset$ there is a maximal element w.r.t. to the inclusion $\subseteq$ order. Every such maximal element is a prime ideal.

<u>Proof.</u> $\{0\} \in \Gamma. \Rightarrow \Gamma \neq \emptyset$. We show that the condition of the Zorn lemma are satisfied: if $L \subseteq \Gamma$ is linearly ordered then $\bigcup L \in \Gamma$ since it is an ideal (proven in Lect. 4) and $(\bigcup L) \cap \mathcal{S} = \emptyset$.

It remains to show that $\forall$ maximal element $\mathcal{J}$ in Γ is a prime ideal. Let $a, b \notin \mathcal{J}$. We need to show that $a \cdot b \notin \mathcal{J}$. $a, b \notin \mathcal{J}$ implies that $\mathcal{A}a + \mathcal{J}$ and $\mathcal{A}b + \mathcal{J}$ are ideals both $\supsetneq \mathcal{J}$. Since $\mathcal{J}$ is maximal in Γ then $(\mathcal{A}a + \mathcal{J}) \cap \mathcal{S} \neq \emptyset \neq (\mathcal{A}b + \mathcal{J}) \cap \mathcal{S}$, i.e. $\exists s, t \in \mathcal{S}$, $c, d \in \mathcal{A}$ s.t. $c \cdot a \equiv s \mod \mathcal{J}$, $d \cdot b \equiv t \mod \mathcal{J} \Rightarrow (c \cdot d) \cdot (a \cdot b) \equiv s \cdot t \mod \mathcal{J}$.
But $s \cdot t \in \mathcal{S}$ and hence, $s \cdot t \notin \mathcal{J}$. $\Rightarrow (c \cdot d) \cdot (a \cdot b) \notin \mathcal{J}. \Rightarrow a \cdot b \notin \mathcal{J}$ (why?) $\square$.

<u>Corollary</u> In a commutative associative algebra $\mathcal{A}$:

$$\bigcap \{\mathcal{J} \subseteq \mathcal{A} \mid \mathcal{J} \text{ is a prime ideal}\} = \{a \in \mathcal{A} \mid a \text{ is nilpotent i.e. } \exists n \in \mathbb{N} \text{ s.t. } a^n = 0\}$$

<u>Proof.</u> $\subseteq$) Let $a^n \neq 0$ $\forall n \in \mathbb{N}$. Then $\mathcal{S} = \{1, a, a^2, \ldots\}$ is a multiplicative. $\Rightarrow \exists \mathcal{J}$-prime, $a \notin \mathcal{J}. \Rightarrow a \notin \bigcap\{\mathcal{J}\text{-prime}\}$. $\supseteq$) Let a be nilpotent. Then $[a] \in \mathcal{A}/\mathcal{J}$ is nilpotent. If $\mathcal{J}$-prime $\Rightarrow [a] = 0$ (why?). $\Rightarrow a \in \mathcal{J}$ $\forall \mathcal{J}$-prime. $\square$

<u>Corollary</u> In a unital commutative associative algebra $\mathcal{A}$ every maximal ideal is prime

Closed ideals in C^*-algebras. Quotient of C^*-algebra.

<u>Theorem 5.2.</u> Let A be a C^*-algebra (possibly nonunital, noncommutative) and let $J \subseteq A$ be a closed two-sided ideal. Then J is self-adjoint, i.e. $a \in J \Rightarrow a^* \in J$. Hence, A/J has a structure of a Banach $*$-algebra. Furthermore, A/J is a C^*-algebra.

<u>Lemma 5.3.</u> Let A be a unital C^*-algebra, $a \in A$, $a \neq 0$. Then $\forall \varepsilon > 0 : \exists (a^*a + \varepsilon.1)^{-1} \in A$

$$a \cdot [(a^*a + \varepsilon.1)^{-1} \cdot a^*a - 1] \xrightarrow[\varepsilon \to 0]{} 0 .$$

<u>Proof.</u> $\sigma_A(a^*a) \subseteq [0,\infty) \Rightarrow \exists (a^*a + \varepsilon.1)^{-1}$. $\|a \cdot [(a^*a + \varepsilon.1)^{-1} a^*a - 1]\|^2 \overset{\text{check!}}{=}$ $= \|(a^*a) \cdot [(a^*a + \varepsilon.1)^{-1} a^*a - 1]^2\|$. By Proposition 4.13:

$$\|(a^*a) \cdot [(a^*a + \varepsilon.1)^{-1} a^*a - 1]^2\| = \sup_{z \in \sigma_A(a^*a)} |z| \left(\frac{|z|}{|z| + \varepsilon} - 1 \right)^2 \xrightarrow[\varepsilon \to 0]{(\text{why?})} 0 . \square$$

<u>Proof of Theorem 5.2 for unital algebras.</u> Why J is self-adjoint?
Let $a \in J$. $\Rightarrow a^*a \in J$. $\Rightarrow (a^*a + \varepsilon.1)^{-1} \cdot a^*a \in J$.

$$\Rightarrow \|a^* - \underbrace{(a^*a) \cdot (a^*a + \varepsilon.1)^{-1} \cdot a^*}_{\in J}\| = \|a - a \cdot (a^*a + \varepsilon.1)^{-1} \cdot (a^*a)\| \xrightarrow[\varepsilon \to 0]{} 0 .$$

Since J-closed $\Rightarrow a^* \in J$.

C^*-identity in A/J? Enough to check: $\|[a]\|^2 \leq \|[a^*a]\|$?
Recall $\|[a]\| = \inf_{c \in J} \|a + c\|$. Let $c \in J$ and set $u_\varepsilon := (c^*c + \varepsilon.1)^{-1} \cdot c^*c \in J$.
Then $u_\varepsilon^* = u_\varepsilon$ and $\|c \cdot (1 - u_\varepsilon)\| \xrightarrow[\varepsilon \to 0]{} 0$.
By Prop. 4.13 $\Rightarrow u_\varepsilon \geq 0$, $1 - u_\varepsilon \geq 0$, $\|1 - u_\varepsilon\| \leq 1$. Then:

$$\|[a]\|^2 \leq \lim_{\varepsilon \to 0} \|a - a \cdot u_\varepsilon\|^2 = \lim_{\varepsilon \to 0} \|(a - a \cdot u_\varepsilon)^* \cdot (a - a u_\varepsilon)\|$$
$$= \lim_{\varepsilon \to 0} \|(1 - u_\varepsilon)^* a^*a (1 - u_\varepsilon)\| \quad \boxed{\text{bounded in norm}}$$
$$= \lim_{\varepsilon \to 0} \|(1 - u_\varepsilon)^* a^*a (1 - u_\varepsilon) + (1 - u_\varepsilon)^* \cdot \underbrace{c \cdot (1 - u_\varepsilon)}_{\to 0,\ \varepsilon \to 0}\|$$
$$= \lim_{\varepsilon \to 0} \|(1 - u_\varepsilon)^* (a^*a + c)(1 - u_\varepsilon)\| \leq \|a^*a + c\| .$$
$$\Rightarrow \|[a]\|^2 \leq \inf_{c \in J} \|a^*a + c\| = \|[a^*a]\| . \square$$

Closed ideals in commutative C^*-algebras

Theorem 5.4. Let A be a unital commutative C^*-algebra.
(a) Every closed prime ideal in A is maximal.
(b) If $J \subsetneq A$ is a closed proper ideal then

$$J = \bigcap \{ J' \subsetneq A \mid J \subseteq J',\ J' \text{ is maximal ideal} \}.$$

Proof. (a) Let J is a closed prime ideal. By Observation 5.1 A/J has no zero divisors and by Theorem 5.3 A/J is a unital commutative C^*-algebra.
We shall prove then that $A/J \cong \mathbb{C}$. Let $a \in A/J$ and $\sigma_A(a)$ has at least two points $x \neq y$. $\exists f, g \in C(\mathbb{C})$ s.t. $f(x) \neq 0 \neq g(y)$ but $f \cdot g = 0$ (why?).
Then $f(a) \neq 0 \neq g(a)$ and $f(a) \cdot g(a) = 0$. Hence $\forall a \in A$: $|\sigma_A(a)| = 1$.
$\Rightarrow \forall a \in A\ \exists z \in \mathbb{C}$: $\sigma_A(a - z.1) = \{0\}$. By Prop. 4.13 $\Rightarrow a = z.1$.
(b) Obviously, $J \subseteq \bigcap \{ J' \subsetneq A \mid J \subseteq J',\ J' \text{ is maximal ideal} \}$. $\supseteq$?
A/J is a unital, commutative C^*-algebra.

$$\{ J' \subsetneq A \mid J \subseteq J',\ J' \text{ is maximal ideal} \} = \{ \pi^{-1}(J'') \mid J'' \subsetneq A/J \text{ is maximal ideal} \}$$

$$\begin{aligned}
\Rightarrow\ & \bigcap \{ J' \subsetneq A \mid J \subseteq J',\ J' \text{ is maximal ideal} \} \\
&= \bigcap \{ \pi^{-1}(J'') \mid J'' \subsetneq A/J \text{ is maximal ideal} \} \\
&= \pi^{-1}\Big(\underbrace{\bigcap \{ J'' \subsetneq A/J \mid J'' \text{ is maximal ideal} \}}_{= \text{Kernel of the Gelfand map} = 0 \text{ for } C^*\text{-algebras}} \Big) = \pi^{-1}\{0\} = J.
\end{aligned}$$

$\square$

Algebraic sets and the topology that they generate

Let A be a unital commutative algebra

$X(A)$ = set of charactes,

$X_m(A) := \{ J' \subsetneq A \mid J' \text{ is a maximal ideal} \}$ - "maximal spectrum"

$X_p(A) := \{ J' \subsetneq A \mid J' \text{ is a prime ideal} \}$ - "prime spectrum"

Natural embeddings:

$$\begin{array}{ccccc}
\chi & \mapsto & \operatorname{Ker} \chi & & \\
X(A) & \hookrightarrow & X_m(A) & \hookrightarrow & X_p(A) \\
 & & J' & \longmapsto & J'
\end{array}$$

We shall consider:

$$X(A) \subseteq X_m(A) \subseteq X_p(A) \text{ under the above injections.}$$

Our results up to now: $X(A) = X_m(A)$ – for Banach algebras.
$X_m(A) = X_p(A)$ – for C^*-algebras

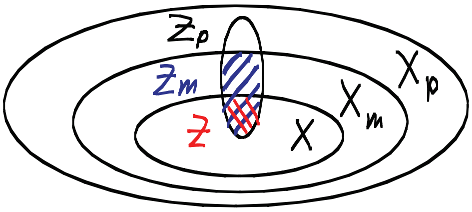
Introduce: for a proper ideal $J \subseteq A$

$$Z(J) := \{ \chi \in X(A) \mid \chi|_J = 0 \}$$

$$Z_\xi(J) := \{ J' \in X_\xi(A) \mid J \subseteq J' \}, \quad \xi = \text{"m"}, \text{"p"}.$$

and set $\quad Z(A) = \emptyset, \quad Z_\xi(A) = \emptyset \quad (\xi = \text{"m"}, \text{"p"})$

Thus: $Z(J) = Z_p(J) \cap X(A)$,
$Z_m(J) = Z_p(J) \cap X_m(A)$.



Maps in the inverse direction (from subsets to ideals):

$$I(S) := \bigcap_{\chi \in S} \operatorname{Ker} \chi, \quad \text{for } S \subseteq X(A);$$

$$I_\xi(S) := \bigcap_{J' \in S} J', \quad \text{for } S \subseteq X_\xi(A), \quad \xi = \text{"m"}, \text{"p"}.$$

$$I(\emptyset) = X(A), \quad I_\xi(\emptyset) = X_\xi(A) \quad \text{for } \xi = \text{"m"}, \text{"p"}.$$

Notation: if not otherwise specified $X_\xi(A)$, $Z_\xi(J)$, $I_\xi(S)$ will denote any of the three cased $X(A)$, $Z(J)$, $I(S)$ or $\xi = \text{"m"}$ and "p".

Sets of a form $Z_\xi(J)$ are called algebraic sets (w.r.t. A).

Proposition 5.5 (a) $J_1 \subseteq J_2 \Rightarrow Z_\xi(J_2) \subseteq Z_\xi(J_1)$;
$S_1 \subseteq S_2 \Rightarrow I_\xi(S_2) \subseteq I_\xi(S_1)$; $\quad J \subseteq I_\xi(Z_\xi(J))$; $\quad S \subseteq Z_\xi(I_\xi(S))$.

(b) In a unital commutative C^*-algebra: $I(Z(J)) = \overline{J}$.

(c) $\forall J_1, J_2$ - ideals: $Z_\xi(J_1 \cap J_2) = Z_\xi(J_1) \cup Z_\xi(J_2)$

(d) $\forall \{J_q\}_{q \in Q}$ - family of ideals: $Z_\xi\left(\bigvee_q J_q\right) = \bigcap_{q \in Q} Z_\xi(J_q)$

where: $\displaystyle \bigvee_{q \in Q} J_q = \bigcap \{ J' \subseteq A \mid J' \text{- ideal}, \ J_q \subseteq J' \ (\forall q \in Q) \}$.

<u>Lemma 5.6</u> Let $\{\mathcal{I}_q\}_{q\in Q}$ be a family of ideals in $\mathcal{A}$ and set $\mathcal{I}_{q_1,\dots,q_n} := \mathcal{I}_{q_1} + \dots + \mathcal{I}_{q_n}$. Then:

$$\bigvee_{q\in Q} \mathcal{I}_q = \bigcup_{n=1}^{\infty} \bigcup_{q_1,\dots,q_n\in Q} \mathcal{I}_{q_1,\dots,q_n} . \qquad (*)$$

<u>Hint:</u> Show that $\mathcal{I}_{q_1,\dots,q_n}$ is an ideal, $\mathcal{I}_{q_1,\dots,q_n} = \bigvee_{k=1}^{n} \mathcal{I}_{q_k}$; then show that r.h.s. of $(*)$ is an ideal, and in fact, the lowest ideal containing all $\mathcal{I}_q$. $\square$

<u>Proof of Proposition 5.5.</u> (a) is left for an exercise.

(b) $\mathcal{I}(\mathcal{Z}(\mathcal{I}))$ is always closed ideal as intersection of closed ideals (lect. 4). Since characters on C^*-algebras are continuous: $\mathcal{Z}(\mathcal{I}) = \mathcal{Z}(\bar{\mathcal{I}})$.
By Theorem 5.4 (b): $\bar{\mathcal{I}} = \mathcal{I}(\mathcal{Z}(\bar{\mathcal{I}})) = \mathcal{I}(\mathcal{Z}(\mathcal{I}))$.

It is enough to check (c) and (d) for $\xi =$ "p".

(c) $\forall \mathcal{J}' \in X_p(\mathcal{A})$: $\mathcal{J}_1 \subseteq \mathcal{J}'$ or $\mathcal{J}_2 \subseteq \mathcal{J}' \Rightarrow \mathcal{J}_1 \cap \mathcal{J}_2 \subseteq \mathcal{J}'$. $\Rightarrow$ "$\supseteq$".

$\subseteq$) Assume the contrary: $\exists \mathcal{J}'$-prime ideal s.t. $\mathcal{J}_1\cap\mathcal{J}_2 \subseteq \mathcal{J}'$ but $\mathcal{J}_k \not\subseteq \mathcal{J}'$ $(k=1,2)$.
Then: $\exists a_k \in \mathcal{J}_k$ s.t. $a_k \notin \mathcal{J}'$ $(k=1,2)$. $\Rightarrow a_1\cdot a_2 \in \mathcal{J}_k$ $(k=1,2)$. $\Rightarrow a_1 \cdot a_2 \in \mathcal{J}_1\cap\mathcal{J}_2 \subseteq \mathcal{J}'$.
But $\mathcal{J}'$ is prime and $a_1\cdot a_2 \in \mathcal{J}'$. $\Rightarrow a_1 \in \mathcal{J}'$ or $a_2 \in \mathcal{J}'$ – contradiction.

(d) $\forall \mathcal{J}'$-prime: $\mathcal{J}' \in \mathcal{Z}_p(\bigvee_q \mathcal{J}_q) \Leftrightarrow \bigvee_q \mathcal{J}_q \subseteq \mathcal{J}' \Leftrightarrow \forall q\in Q: \mathcal{J}_q \subseteq \mathcal{J}'$
$\Leftrightarrow \forall q \in Q: \mathcal{J}' \in \mathcal{Z}_p(\mathcal{J}_q) \Leftrightarrow \mathcal{J}' \in \bigcap_q \mathcal{Z}_p(\mathcal{J}_q)$. $\square$

<u>Corollary 5.7</u> The system of algebraic sets $\{\mathcal{Z}_\xi(\mathcal{J}) \mid \mathcal{J}\text{-ideal in } \mathcal{A}\}$ defines a topology on $X_\xi(\mathcal{A})$ in terms of closed sets, i.e. $\mathcal{Z}_\xi(\mathcal{J})$ are the closed sets of this topology.

<u>Exercise:</u> $\forall S \subseteq X_\xi(\mathcal{A})$: $\bar{S} = \mathcal{Z}_\xi(\mathcal{I}_\xi(S))$.

<u>Hint:</u> If $S' = \mathcal{Z}_\xi(\mathcal{J}') \Rightarrow \mathcal{I}_\xi(S') = \mathcal{I}_\xi(\mathcal{Z}_\xi(\mathcal{J}')) \supseteq \mathcal{J}'$.
$\Rightarrow \mathcal{Z}_\xi(\mathcal{I}_\xi(S')) \subseteq \mathcal{Z}_\xi(\mathcal{J}') = S'$. $\Rightarrow \mathcal{Z}_\xi(\mathcal{I}_\xi(S')) = S'$.

<u>Remark.</u> This topology in the case of finitely generated algebras (the case in algebraic geometry) is called Zariski topology (топология на Зариски) on $X_p(\mathcal{A})$. It is non-Hausdorff and even not T_1 (in general): $\overline{\{\mathcal{J}\}} = \{\mathcal{J}' \in X_p(\mathcal{A}) \mid \mathcal{J} \subseteq \mathcal{J}'\}$, $\mathcal{J} \in X_p(\mathcal{A})$.
But this depends of course on the the algepa $\mathcal{A}$ and we shall see below that the above topology for a unital commutative C^*-algebra is always compact Hausdorff.

Note the topologies of algebraic sets on $X(\mathcal{A})$ and $X_m(\mathcal{A})$ coincide with the induced ones by the topology of algebraic sets on $X_p(\mathcal{A})$ (why?).

<u>Proposition 5.8</u> Let $\mathcal{A}$ be a unital commutative algebra. Then $X_m(\mathcal{A})$ and $X_p(\mathcal{A})$ are compact spaces.

<u>Proof.</u> Recall $(Y,\mathcal{T})$ is compact topological space $\Leftrightarrow$ $\forall \mathcal{O} \subseteq \mathcal{T}$ s.t. $Y = \bigcup \mathcal{O}$ $\exists\, \mathcal{O}_0 \underset{\text{fin.}}{\subseteq} \mathcal{O}$ s.t $Y = \bigcup \mathcal{O}_0$. In terms of the complementary closed sets $\mathcal{F} = \{F \subseteq Y \mid Y \setminus F \in \mathcal{O}\}$: if $\bigcap \mathcal{F} = \emptyset \Rightarrow \exists\, \mathcal{F}_0 \underset{\text{fin}}{\subseteq} \mathcal{F}$ s.t. $\bigcap \mathcal{F}_0 = Y$. Thus, the compactness of $(Y,\mathcal{T})$ is equivalent to:

$\forall \mathcal{F}$-family of closed sets s.t. $\forall \mathcal{F}_0 \underset{\text{fin.}}{\subseteq} \mathcal{F}$, $\bigcap \mathcal{F}_0 \neq \emptyset \Rightarrow \bigcap \mathcal{F} = \emptyset$.

Let us check the above criterion for $X_\xi(\mathcal{A})$, $\xi =$ "m" and "p".

Let $\mathcal{F} = \{Z_\xi(\mathcal{J}_q) \mid q \in Q\}$ – family of algebraic sets s.t. $\forall n \in \mathbb{N}$, $\forall q_1, \dots, q_n \in Q$: $\bigcap_{k=1}^{n} Z_\xi(\mathcal{J}_{q_k}) \neq \emptyset$. Why $\bigcap_q Z_\xi(\mathcal{J}_q) \neq \emptyset$?

Note: $\bigcap_{k=1}^{n} Z_\xi(\mathcal{J}_{q_k}) = Z_\xi(\mathcal{J}_{q_1,\dots,q_n}) \neq \emptyset \overset{\text{why?}}{\Leftrightarrow} \mathcal{J}_{q_1,\dots,q_n}$ - proper.

Similarly, $\bigcap_q Z_\xi(\mathcal{J}_q) = Z_\xi(\bigvee_q \mathcal{J}_q) \neq \emptyset \Leftrightarrow \bigvee_q \mathcal{J}_q$ - proper

By Lemma 5.6: $\bigvee_q \mathcal{J}_q = \bigcup_n \bigcup_{q_1,\dots,q_n} \mathcal{J}_{q_1,\dots,q_n}$ and recall

$\mathcal{J}$ is proper $\Leftrightarrow$ $\mathcal{J} \subseteq N.I.E.(\mathcal{A})$ for unital commutative algebras.

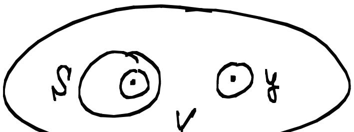
Thus, $\bigvee_q \mathcal{J}_q$ is proper. $\Rightarrow \bigcap \mathcal{F} = Z_\xi(\bigvee_q \mathcal{J}_q) \neq \emptyset$. $\square$

Is this true in general for $Z(\mathcal{J})$?

Additional facts from topology

1.) Let Y be compact Hausdorff space. Then Y is "regular", i.e $\forall\, \emptyset \neq S \underset{closed}{\subseteq} Y$, $\forall y \in Y \setminus S$, $\exists\, \mathcal{U}, V \underset{open}{\subseteq} Y$, $S \subseteq \mathcal{U}$, $y \in V$, $\mathcal{U} \cap V = \emptyset$.

Proof. Y is Hausdorff $\Rightarrow$

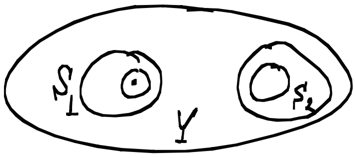


$\forall t \in S$, $\exists\, \mathcal{U}_t, V_t \underset{open}{\subseteq} Y$, $t \in \mathcal{U}_t$, $y \in V_t$, $\mathcal{U}_t \cap V_t = \emptyset$

$\{\mathcal{U}_t\}_{t \in S}$ - covers S. S is closed $\Rightarrow$ S is compact. $\Rightarrow S \subseteq \mathcal{U}_{t_1} \cup \dots \cup \mathcal{U}_{t_n} =: \mathcal{U}$
$V := V_{t_1} \cap \dots \cap V_{t_n}$. $\square$

2.) Let Y be compact Hausdorff space. Then Y is "normal", i.e $\forall\, \emptyset \neq S_k \underset{closed}{\subseteq} Y$ $(k=1,2)$, $S_1 \cap S_2 = \emptyset$, $\exists\, \mathcal{U}_k \underset{open}{\subseteq} Y$ $(k=1,2)$, $S_k \subseteq \mathcal{U}_k$, $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$.

Proof. Similar:



3.) Urysohn's Lemma (Лема на Урисон). If Y is normal space and S_1, S_2 - as in 2.) then $\exists f \in C(Y)$ s.t. $f|_{S_1} = 0$, $f|_{S_2} = 1$.

Sketch of the proof. (see R. Engelking, "General Topology" Theorem 1.5.10)
Enumerating all rational numbers in $[0,1]$: $r_1, r_2, \dots$ we construct inductively: $V_{r_n} \underset{open}{\subseteq} Y$ s.t. $\overline{V_{r_j}} \subseteq V_{r_k}$ if $r_j < r_k$, $S_1 \subseteq V_{r_1}$, $S_2 \subseteq Y \setminus V_{r_2}$.
Then $f(y) = \inf\{r_k \mid y \in V_{r_k}\}$ for $y \in V_1$ and $f|_{Y \setminus V_1} = 0$. $\square$

4.) Let S be a set, $\Gamma \subseteq Maps(S, \mathbb{C})$. We say that Γ separates the points of S if $\forall x, y \in Y$ $\exists f \in \Gamma$ s.t. $f(x) \neq f(y)$. We say that Γ does not vanish at every point of S if $\forall x \in S$, $\exists f \in \Gamma$, $f(x) \neq 0$.

5.) Stone-Weierstrass Theorem (Теорема на Стоун-Ваерщрас).
Let Y be compact Hausdorff space, $\mathcal{A}_0 \subseteq C(Y)$ is subalgebra that separates the points of Y, $\mathcal{A}_0$ does not vanish at every point of Y, selfadjoint: $\mathcal{A}_0 = \mathcal{A}_0^*$. Then $\overline{\mathcal{A}_0} = C(Y)$ in the topology of uniform norm.

Proof Rudin "Principles of Mathematical Analysis", Sects. 7.24 - 7.31.

The C^*-algebra $C(Y)$

Theorem 5.9 Let Y be compact Hausdorff space. Then $C(Y)$ is a unital commutative C^*-algebra and the natural map:

$$Y \to X(C(Y)) : y \mapsto \chi_y \,, \quad \chi_y(f) := f(y) \;\; \forall y \in Y$$

is a homeomorphism.

Proof 1.) For every topological space Y, $C(Y)$ is a closed subspace in Bounded Maps $(Y, \mathbb{C})$:

If $\{f_k\} \subseteq C(Y)$, $f_k \rightrightarrows f$ then $f \in C(Y)$?

If $y = f(x)$, $\varepsilon > 0$: $\exists\, U_x$ - neighbourhood of x, s.t.

$$f(U_x) \subseteq B_\varepsilon(y) := \{y' : |y' - y| < \varepsilon\} \;?$$

$$|f(x') - f(x)| < \underbrace{|f(x') - f_k(x')|}_{<\varepsilon/3} + \underbrace{|f_k(x') - f_k(x)|}_{<\varepsilon/3} + \underbrace{|f_k(x) - f(x)|}_{<\varepsilon/3}$$

Then for $k \geq N(\varepsilon/3)$ $\left(|f_k(x') - f(x)| \leq \|f_k - f\| < \frac{\varepsilon}{3} \right)$, $U_x := f_k^{-1}(B_{\varepsilon/3}(y))$

Thus, $C(Y)$ is a unital, commutative, C^*-algebra.

2.) Why $Y \to X(C(Y)) : y \mapsto \chi_y$ is a bijection (биекция)?

It is injection: by Urysohn's Lemma - if $y_1 \neq y_2$ $\exists f \in C(Y)$: $f(y_1) \neq f(y_2)$. $\Rightarrow \chi_{y_1} \neq \chi_{y_2}$.

It is surjection: Let $\chi \in X(C(Y))$, we need to show that $\cap \{f^{-1}(0) \mid f \in \operatorname{Ker} \chi\} \neq \emptyset$.

Indeed: if $y \in \cap\{f^{-1}(0) \mid f \in \operatorname{Ker}\chi\}$ then $\chi_y(f) = f(y) = 0$ $\forall f \in \operatorname{Ker}\chi$ and $\Rightarrow \chi_y = \chi$ since $\chi_y(1) = \chi(1) = 1$ (and $\ker \chi_y \subseteq \ker \chi$).

We shall prove more general statement:

3.) Let $J \subseteq C(Y)$ - proper ideal. Then $Z_Y(J) := \cap\{f^{-1}(0) \mid f \in J\} \neq \emptyset$ (it is also closed as intersection of closed subsets).

Proof We use the criterion used in Proposition 5.8: we show that $\forall f_1, \dots, f_n \in J$: $\bigcap_{k=1}^{n} f_k^{-1}(0) \neq \emptyset$ and then it will follow that $\cap\{f^{-1}(0) \mid f \in J\} \neq \emptyset$.

Assume the contrary, $\exists f_1, \dots, f_n \in J$ s.t. $\bigcap_{k=1}^{n} f_k^{-1}(0) = \emptyset$. $\Rightarrow |f_1(y)|^2 + \dots + |f_n(y)|^2 \neq 0$ $(\forall y \in Y)$. $\Rightarrow (|f_1|^2 + \dots + |f_n|^2)^{-1} \in C(Y)$ (why? - Y is compact).

But $\sum_{k=1}^{n} |f_k|^2 = \sum_{k=1}^{n} f_k^* f_k \in J$. $\Rightarrow$ contradiction J - proper.

4.) Set in addition $\mathcal{Z}_Y(\mathcal{A}) = \emptyset$. For $\forall S \subseteq Y$: $\mathcal{I}_Y(S) := \{f \in C(Y) \mid f|_S = 0\}$.

Then : $\forall$ closed ideal $\mathcal{J} \subseteq \mathcal{A}$: $\mathcal{I}_Y(\mathcal{Z}_Y(\mathcal{J})) = \mathcal{J}$; (1)

$\forall$ closed $S \subseteq Y$: $\mathcal{Z}_Y(\mathcal{I}_Y(S)) = S$. (2)

Note first: under the identification $Y \xrightarrow{\sim} X(C(Y))$: $\mathcal{Z}_Y(\mathcal{J}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{J})$.

Then (1) follows by Theorem 5.3 (b).

(2) : $\supseteq$) - by the definitions ; $\subseteq$) assume $\exists y \in \mathcal{Z}_Y(\mathcal{I}_Y(S))$, $y \notin S$.

By Urysohn's Lemma $\exists f \in C(Y)$ s.t. $f(y) = 1$, $f|_S = 0$. This contradicts to $f \in \mathcal{I}_Y(S)$, $y \in \mathcal{Z}_Y(\mathcal{I}_Y(S))$ i.e. $f(y) = 0$.

5.) Thus, we have one-to-one correspondence between closed sets and closed ideals :

$$\begin{tikzcd}
\left\{\begin{matrix}\text{closed subsets}\\ \text{in } Y\end{matrix}\right\} \arrow[r, shift left, "\mathcal{I}_Y"] \arrow[rr, bend right, leftrightarrow, "\text{under the bijection } Y \xrightarrow{\sim} X(C(Y))"'] & \left\{\begin{matrix}\text{closed ideals}\\ \text{in } C(Y)\end{matrix}\right\} \arrow[l, shift left, "\mathcal{Z}_Y"] \arrow[r, shift left, "\mathcal{Z}"] & \left\{\begin{matrix}\text{closed subsets}\\ \text{in } X(C(Y))\end{matrix}\right\} \arrow[l, shift left, "\mathcal{I}"]
\end{tikzcd}$$

$\Rightarrow Y \xrightarrow{\sim} X(C(Y))$ is homeomorphism. $\square$

Morphisms and continuous maps.

This relation is most transparent in $X(\mathcal{A})$:

Let $\mathcal{A}$ and $\mathcal{B}$ be unital commutative algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of unital algebras. Define :

$$\varphi^* : X(\mathcal{B}) \to X(\mathcal{A}) : \chi \mapsto \chi \circ \varphi$$

$$\begin{tikzcd}
 & \mathbb{C} & \\
\mathcal{A} \arrow[ur, "\varphi^*(\chi)"] \arrow[rr, "\varphi"'] & & \mathcal{B} \arrow[ul, "\chi"']
\end{tikzcd}$$

$$(\varphi^*)^{-1}\mathcal{Z}(\mathcal{J}) = \{\chi \in X(\mathcal{B}) \mid \varphi^*(\chi) \in \mathcal{Z}(\mathcal{J})\} = \{\chi \in X(\mathcal{B}) \mid \varphi^*(\chi)|_{\mathcal{J}} = 0\}$$

$$= \{\chi \in X(\mathcal{B}) \mid \chi|_{\varphi(\mathcal{J})} = 0\} = \{\chi \in X(\mathcal{B}) \mid \varphi(\mathcal{J}) \subseteq \ker \chi\} = \mathcal{Z}(\mathcal{B} \cdot \varphi(\mathcal{J})).$$

$(\varphi^*)^{-1}\mathcal{Z}(\mathcal{J}) = \mathcal{Z}(\mathcal{B} \cdot \varphi(\mathcal{J}))$ - inverse image of an algebraic set is algebraic.

$\Rightarrow \varphi^*$ is continuous.

Corollary 5.10 For a unital, commutative C^*-algebra A $\hat{a} \in C(X(A))$ $\forall a \in A$, i.e. the Gelfand map $a \mapsto \hat{a} : A \to C(X(A))$.

Proof Enough to prove for $a = a^*$. By Prop. 4.13 : $C(\sigma_A(a)) \cong \overline{\mathbb{C}[a]}$.

$$\begin{tikzcd}
\overline{\mathbb{C}[a]} \arrow[r, phantom, "\cong"] \arrow[d, hook, "\iota"'] & C(\sigma_A(a)) \arrow[dl, "\iota_1"] \\
A &
\end{tikzcd}
\quad \text{induces} \quad
\begin{tikzcd}
X(\overline{\mathbb{C}[a]}) \arrow[r, phantom, "\cong"] & X(C(\sigma_A(a))) \arrow[r, phantom, "\cong"] & \sigma_A(a) \\
X(A) \arrow[u, "\iota^*"] \arrow[ur, "\iota_1^*"] \arrow[urr, "? = \hat{a}"'] & &
\end{tikzcd}$$

we claim

check: $(\chi \circ \iota)(f(a)) = \chi'(f)$, $\chi' = \chi_z$

$$\begin{tikzcd}
\chi \circ \iota \arrow[r, leftrightarrow] & \chi' \arrow[r, leftrightarrow] & z = \chi(a) = \hat{a}(\chi) \\
\chi \arrow[u, "\iota^*"] \arrow[ur, mapsto, "\iota_1^*"] \arrow[urr, mapsto, "? = ?"'] & &
\end{tikzcd}$$

$$\chi'(f) = (\chi \circ \iota)(f(a)) = \chi(f(a)) = \widehat{f(a)}(\chi) = (f \circ \hat{a})(\chi) = f(\hat{a}(\chi)) = f(\chi(a))$$
$$\parallel$$
$$\chi_z(f) = f(z) \quad \forall f \in C(\sigma_A(a)) \Rightarrow z = \chi(a) = \hat{a}(\chi)$$

Hence, $\hat{a}$ is continuous as composition of continuous maps. $\square$

The topology on $X(A)$ is Hausdorff for a unital, commutative C^*-algebra A

Denote $\Theta : A \to C(X(A)) : a \mapsto \hat{a}$ - the Gelfand map.

Let $\chi_1, \chi_2 \in X(A)$, $\chi_1 \neq \chi_2$. $\Rightarrow \exists a \in A : \chi_1(a) \neq \chi_2(a)$, i.e. $\hat{a}(\chi_1) \neq \hat{a}(\chi_2)$.

$\Rightarrow \Theta(A)$ separates the points of $X(A)$. It also does not vanish at every point of A since $\hat{1}(\chi) = \chi(1) = 1$ $\forall \chi \in X(A)$.

In order to apply the Stone-Weierstrass theorem we only need to show that $X(A)$ is Hausdorff (we know - it is compact).

To this end: let $\chi_1 \neq \chi_2$, $\hat{a}(\chi_1) \neq \hat{a}(\chi_2)$ as above. $\exists V_1, V_2 \underset{\text{open}}{\subseteq} \mathbb{C}$ s.t. $\hat{a}(\chi_k) \in V_k$ $(k=1,2)$, $V_1 \cap V_2 = \emptyset$. $\Rightarrow U_k := \hat{a}^{-1}(V_k) \underset{\text{open}}{\subseteq} X(A)$

$\chi_k \in U_k$ $(k=1,2)$, $U_1 \cap U_2 = \emptyset$.

Corollary 5.11. If $\mathcal{A}$ is unital, commutative C^*-algebra then $X(\mathcal{A})$ is a compact Hausdorff space and the Gelfand map $\mathcal{A} \to C(X(\mathcal{A})) : a \mapsto \hat{a}$ is isomorphism.

Summary

For a unital, commutative C^*-algebra $\mathcal{A}$:

1.) $X(\mathcal{A})$ - compact Hausdorff w.r.t. topology:

$$\left\{ \begin{matrix} \text{closed subsets} \\ \text{of } X(\mathcal{A}) \end{matrix} \right\} \underset{Z}{\overset{I}{\rightleftarrows}} \left\{ \begin{matrix} \text{closed ideals} \\ \text{of } \mathcal{A} \end{matrix} \right\}$$

- inverse maps.

2.) $\mathcal{A} \cong C(X(\mathcal{A}))$ under the Gelfand map

3.) For Y - compact Hausdorff space: $\underset{\text{homeomorphism.}}{Y \xrightarrow{\sim} X(C(Y))} : y \mapsto \chi_y$

4.) $\forall\, \mathcal{A}, \mathcal{B}$ - unital, commutative C^*-algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ - morphism of unital $*$-algebras: $\varphi^* : X(\mathcal{B}) \underset{\text{continuous}}{\longrightarrow} X(\mathcal{A}) : \chi \mapsto \chi \circ \varphi$.

$(\varphi \circ \psi)^* = \psi^* \circ \varphi^* \quad (\psi : \mathcal{B} \to \mathcal{C})$

5.) $\forall\, F : Y \to Y'$ continuous between compact Hausdorff spaces $F^* : C(Y') \to C(Y) : f \mapsto f \circ F$ - morphism of unital C^*-alg.